THREE IMPOSSIBLES THEOREMS

Cédric Villani – The world around us contains many strange things, in mathematics too, impossible things even. Sometimes we teach them and draw conclusions from them. The most surprising statement in all of mathematics is said to be the Banach-Tarski paradox: if I take a ball, I can cut it up in a finite number of pieces in such a way that I can move the pieces around and stick them together again to create a bust of Poincaré, or a life-size statue of an elephant. Obviously, that can't be right! It would be great if we could do that with a solid gold ball… Naturally, we can't actually produce these pieces… unless we use the powerful and controversial axiom of choice. I taught my students this paradox, at École normale supérieure in Lyon, to warn them against the axiom of choice.

Another impossible statement is the Nash-Kuiper theorem. That one doesn't use the axiom of choice, but it's just as paradoxical. Let's say that I now want to put my ball into a small matchbox. Well, I can, says Nash, I can fold it over on itself so that it fits in the matchbox, without denting it or changing its symmetry. A tiny ant on the surface of the ball wouldn't even notice the change. This makes no sense! We're used to thinking that a sphere is rigid because of its curvature. That's another theorem I worked through with students and we demonstrated it again. It teaches us that appearances are deceptive and that rigidity is a subtle concept that depends on regularity. It's a wonderful construct, you can actually see it, it's extremely beautiful.

The third statement I'll mention is the one I presented in a Bourbaki seminar, discovered in 1993 by the unpredictable researcher Vladimir Scheffer. Imagine a balmy summer afternoon, a peaceful lake, not a breath of wind and suddenly, the water starts to bubble and splash everywhere… Then everything stops… and there's still no wind. It's as if energy had been generated out of nothing only to go back to nothingness! That's what Scheffer's theorem tells us. Using the good old Euler equation from 1755, he builds a fluid that contravenes the law of energy conservation. We don't really know what the right answer is, how we should react to the Scheffer paradox. In this case we don't know whether we have smooth or non-smooth solutions, or even if that is where the solution lies. What conclusions can we draw? For the moment at least, we can conclude that we understand very little about fluid mechanics and about many familiar objects around us!

3min 09sec