MY ENCOUNTERS WITH JACQUES

Alain Connes – My first encounter with Jacques Dixmier goes back to 1971. I had been invited in the summer of 1971 to the Battele Institute in Seattle and on the way, in Princeton, I had by chance bought a Lecture Notes. The author was a Japanese mathematician: Takesaki, and he was expounding the work of Tomita, another Japanese mathematician. While, with my wife, we were travelling by train through Canada and I had become fascinated by this book, trying to read it. And when I arrived at the conference in Seattle, I discovered that Takesaki was one of the lecturers. I found that amazing, and when I saw this chance, I decided on the spot to go only to his talks and to work hard on that topic. When I got back to France, I went, in September, to attend the Dixmier seminar in Paris. And there, again by chance, I took a paper on the Araki-Woods work which Dixmier was seeking to expound in his seminar, and, on the train ride to the suburbs on my way home, I realized that there was a formidable link between the two theories. At this point I sent Dixmier a one page letter explaining my finding and he replied immediately asking for more details. I replied two days later with a four page letter and it is there that our interaction began; we met in his office and he had just one instruction: "Go Ahead!"

Jacques Dixmier –The second letter he sent me, I remember, was proving unexpected and obviously important results, and I was amazed to see it done in 4 pages, that’s why I told him "Go ahead! " and then, the four pages became the more than hundred pages of your thesis.

AC – Around 1985 another episode of our interaction took place in IHÉS, Jacques had discovered in the 50’s an exotic trace for operators in Hilbert space.

JD – I told you: I am surprised that this construction of mine did not help to construct counter-examples, because what I had found which you called "exotic" was rather for me a monstrosity. And I still remember Alain telling me "but this is exactly what I need! "

AC – Yes, in fact this construction is now called the Dixmier trace and very often, convergence eliminates the involved choices that could seem pathological. It is a kind of measurability. And then there is a truly remarkable fact: this construction covers all the known examples of integrals in mathematics!

JD – Here you exaggerate somewhat!

AC – No, I do not exaggerate, usually in maths when one writes $\int f(x)dx$, the integral sign cannot be dissociated from what one calls the measure which one denotes dμ(x), it is only the whole package that makes sense… But the construction of Jacques gives an independent meaning to the integral so that the integral sign and the dx have independent meanings… Another key point is that, in their doings, physicists have noticed the many occurrences of logarithmic divergences. And the construction of Jacques shows that the coefficient of the logarithmic divergence is a trace, and gives a precise mathematical status to these divergences.

JD – Ah, if Leibniz had known this! Ooh la la!

AC – Yes, but it is precisely here that the points of view of Leibniz and Newton differ substantially! What the Dixmier trace together with the quantum formalism allow one to handle is much closer to Newton's idea of infinitesimals than to Leibniz's. Newton had the idea that infinitesimals ought to be variables rather than numbers. And in mathematics one finds out that a good formulation of the notion of a real variable is as a self-adjoint operator in Hilbert space. This is the only formulation which allows for the coexistence of discrete and continuous variables. And what is amazing is that when one reads Newton's definition of infinitesimal variables one gets exactly the compact operators in Hilbert space. And an infinitesimal can now have an order, it may be of order 1, of order any positive real number, etc. . And it is precisely the case that the Dixmier trace integrates infinitesimals of order at least 1 but gives zero for any infinitesimal of order strictly greater than 1. Thus his trace is a kind of filter which eliminates all quantum details and delivers a classical picture of the result… And this feature played a key role in the new developments!

JD – Developments not due to me!

AC – Thus this second great interaction took place during lunch in IHÉS! By chance! This third and most recent interaction took place six years ago. We were with Danye in the country and received a postcard from Jacques: I have the title of the book: *Let's bother with the boson*! you write it, I will proofread!

JD – It was the time of the discovery of the Higgs boson, not yet found...

AC – At that time I had heard from Étienne Klein an anagram involving the Higgs boson. One side is "le boson scalaire de Higgs" and the other "l'horloge des anges ici-bas." If you shift to commutative and don't take the letters' order into account, the two sentences are exactly the same. We had found a clock decorated with angels, so we sent its picture to Jacques together with the anagram. At that point it was like a joke but gradually the book came into existence. And gradually we added more and more scientific details, in an almost effortless way.

JD – As far as efforts go, I am no longer able to invent mathematics but writing a novel is so much easier…

AC – Finally there is a recent episode, I came to see Jacques one afternoon and showed him the Compte-Rendu note we had just written with Katia Consani, on the arithmetic site. Jacques read the note carefully …

JD – Without understanding much!

AC – Oh, oh, except that he noticed an extraordinary relation with the work he had done on the classification of matroids in the 1960's. He noticed that the space which classifies matroids is the same as the space of points of the arithmetic site! And by looking closer we then understood that this topos is underlying non-commutative geometry! Indeed, in non-commutative geometry, the « point » corresponds to the algebra of compact operators, this algebra has endomorphisms and they define exactly the same topos… And when one looks at this algebra as a sheaf on the topos, the stalks of this sheaf give the matroid classification of Jacques. At the conceptual level this gave us the meaning of the topos we had found with Katia: this topos describes the structure of the point in NCG! This extremely satisfactory result came from the care with which Jacques read our note and made the link with his previous work.